End-Semester Exam ISI Bangalore May 2019

Introduction to Algebraic Geometry 100 Points

Notes.
(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.
(b) There are a total of 120 points in the paper. You will be awarded a maximum of 100.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(c) By default, k denotes an algebraically closed field and $\mathbb{A}_k^n$ is the affine n-space over k while $\mathbb{P}_k^n$ is the projective n-space over k. By default, the polynomial ring of functions on $\mathbb{A}_k^n$ is denoted as $k[x_1, \ldots, x_n]$ while for $n = 1, 2, 3$ we also use the usual notation of x, y, z for the variables.
(d) We will use $\mathcal{V}(-)$ to denote the common zero locus (in suitable affine or projective space) of any collection of polynomials and $\mathcal{I}(-)$ the ideal of functions vanishing on a given subset of affine or projective space.

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1. [16 points] In each case below, give an example of a map of quasi-projective algebraic sets $f\colon X \to Y$ such that the induced map of the ring of regular functions $f^*\colon \mathcal{O}[Y] \to \mathcal{O}[X]$ is an isomorphism while the given additional condition also holds.
   (i) f is surjective, but not injective.
   (ii) f is injective, but not surjective.

2. [16 points] Let $f\colon X \to Y$ be a map of affine algebraic sets. Prove that f has a dense image $\iff$ the induced map of coordinate rings $f^*\colon k[Y] \to k[X]$ is injective.

3. [16 points] Let V denote the union of the x-axis, the y-axis and the z-axis in $\mathbb{A}_k^3$.
   (i) Prove that $\mathcal{I}(V)$ is the ideal (xy, yz, zx) in $k[x, y, z]$.
   (ii) Prove that V is not isomorphic to any quasi-affine algebraic set in $\mathbb{A}_k^2$.

4. [16 points] Consider the regular map $\phi\colon \mathbb{P}^1 \to \mathbb{P}^3$ given by $[a : b] \to [a^3 : a^2b : ab^2 : b^3]$. Let C denote the image of ϕ.
   (i) Prove that C is a closed subset of $\mathbb{P}^3$.
   (ii) Verify that ϕ induces a bijection from $\mathbb{P}^1$ to C.
   (iii) Verify that the inverse map $C \to \mathbb{P}^1$ is regular to deduce that $\phi\colon \mathbb{P}^1 \to C$ is an isomorphism

5. [16 points] Let $k[x, y, z, w]$ denote the homogeneous coordinate ring of $\mathbb{P}^3$. Construct a non-constant regular map from the variety $\mathcal{V}(xy - zw)$ in $\mathbb{P}^3$ to $\mathbb{P}^1$.

6. [20 points] Let A, B be two distinct hyperplanes in $\mathbb{P}^3$ intersecting in a line L. Let M be a line in A different from L. Let $Z = (A \cup B) \setminus M$.

(i) Find the irreducible components of Z.
(ii) Prove that the natural restriction map $R = \mathcal{O}[Z] \longrightarrow \mathcal{O}[A \setminus M] = S$ is injective and identifies elements of R with those elements of S that are constant along L.
(iii) Deduce that $\mathcal{O}[Z]$ is isomorphic to the ring $k[x, xy, xy^2, xy^3, \ldots]$.
(iv) Prove that $\mathcal{O}[Z]$ is not a noetherian ring.

7. [20 points] Let $f\colon X \to Y$ be a map of irreducible quasi-projective varieties such that the image of f is dense in Y.

(i) Prove that there is a natural induced map of function fields $K(Y) \to K(X)$.
(ii) If $K(Y) \to K(X)$ is an algebraic extension, prove that there exist affine open subsets $U \subset X$, $V \subset Y$, with $f(U) \subset V$ such that the coordinate ring $k[U]$ is a free module of finite rank over the coordinate ring $k[V]$.
(iii) Prove that if $[K(X)\colon K(Y)] = d$ in (ii), then the rank of $k[U]$ over $k[V]$ is d and deduce that every fiber of the natural map $U \to V$ has cardinality at most d, (i.e., for every point $q \in V$ there are at most d points in U mapping to it).

(Hint: For (iii), you may use the fact that if R is a k-algebra of finite vector-space dimension d, then R has at most d prime ideals.)